

# THE RING OF FLUXIONS

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**ABSTRACT.** The ring of fluxions (real sequential germs at infinity) provides a rigorous approach to infinitesimals, different from the better-known approach of Abraham Robinson. The basic idea was first espoused in a paper by Curt Schmieden and Detlof Laugwitz published in 1958. Although this ring codifies all the usual intuitive properties of infinitesimals in a very elementary way, its existence has been generally ignored.

## INTRODUCTION

In 1960 Abraham Robinson [6, 7] discovered that a rigorous theory of infinitesimal real numbers can be developed in terms of the existence of nonstandard models of the real number field. Robinson's approach, which depends on the compactness theorem of first-order logic, is not easily accessible to the mathematical community at large.

It is not widely known that two years prior to Robinson's insight, C. Schmieden and D. Laugwitz published a paper [8] in which a more elementary approach to infinitesimals was outlined. This approach is based on the construction of an ordered ring extension  $\mathbb{R}$  of the real ordered field  $\mathbb{R}$ , consisting of germs of real sequences at infinity. In this article, I have taken the liberty to introduce the term *fluxions* for such germs, adapting an obsolete term from [5].

According to [7, p. 2], "G.W. Leibnitz argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with real numbers but which were *to possess the same properties as the latter*." This of course demands not just an ordered ring extension but an ordered field extension, which is what Robinson's method produces. Yet the earlier method of Schmieden and Laugwitz seems to me more generally accessible and intuitively appealing, and the difficulties imposed by having a ring instead of a field in which to house the infinitely small and large quantities are minor.

Although Cauchy is generally credited with the modern epsilon-delta definition of limit, a look at his 1821 *Cours d'analyse* reveals no such

definition stated explicitly. Instead we find the following suggestive statements [1]:

We call a quantity variable if it can be considered as able to take on successively many different values.

When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the limit of all the other values.

When the successive numerical values of the same variable decrease indefinitely, such as to become less than a given number, this number becomes what is called an infinitesimal or an infinitesimal quantity. The limit of this type of variable is zero.

This language seems consistent with the approach to convergence via the ring of fluxions, and suggests [3, 4] that the idea may have been anticipated by Cauchy. Intuitively, a fluxion is merely a sequence of real numbers, with an equality rule that says two sequences define the same fluxion whenever they differ in only a finite number of terms. A fluxion (*e.g.*,  $[1/n]$ ) is infinitesimal when it eventually becomes smaller in magnitude than any given positive real number. Inequalities and real-valued functions of a real variable extend easily to fluxions.

The purpose of this article is to demonstrate how a number of standard results of basic analysis could be formulated and proved using the ring of fluxions. It is far from comprehensive; indeed I was forced to be rather selective of topics in order to keep the article to a reasonable length. Once the ring of fluxions has been precisely defined, and its ordering properties developed, the task of rewriting standard definitions and arguments in the fluxional language is in most cases quite straightforward, with no loss of mathematical rigor. In some instances (*e.g.* uniform continuity in Section 3) concepts may seem more intuitively appealing when expressed in the new language.

## 1. THE ORDERED RING $\dot{\mathbb{R}}$ OF FLUXIONS

Taking the ordered field  $\mathbb{R}$  of ordinary real numbers as given [2], along with its least upper bound property, we wish to construct an ordered ring extension  $\dot{\mathbb{R}}$  of  $\mathbb{R}$  which contains infinitely small and infinitely large quantities. Elements of  $\dot{\mathbb{R}}$  will be called fluxions.

In order to construct the ring  $\dot{\mathbb{R}}$ , consider first the larger structure  $\mathbb{R}^{\mathbb{N}}$  of all sequences  $x: \mathbb{N} \rightarrow \mathbb{R}$  of real numbers. Here,  $\mathbb{N}$  is the set of natural numbers; *i.e.*, positive integers. As usual, given a sequence  $x: \mathbb{N} \rightarrow \mathbb{R}$  in  $\mathbb{R}^{\mathbb{N}}$ , we also write  $(x_n)$  or  $(x_n)_{n \in \mathbb{N}}$  or  $(x_n)_{n=1}^{\infty}$  for it.

The set  $\mathbb{R}^{\mathbb{N}}$  is a commutative ring with the usual ring operations of componentwise addition and componentwise multiplication of sequences:

$$(1.1) \quad (x_n) + (y_n) = (x_n + y_n); \quad (x_n) \cdot (y_n) = (x_n y_n).$$

Let  $Z$  be the set of sequences  $(x_n)$  which are *eventually zero*, meaning that  $x_n$  is non-zero for only finitely many values of  $n$ . In other words,  $(x_n) \in Z$  if and only if there is some natural number  $N$  for which  $x_n = 0$  for all  $n \geq N$ . It is easily checked that  $Z$  is an ideal in the ring  $\mathbb{R}^{\mathbb{N}}$ . We define  $\dot{\mathbb{R}}$  to be the corresponding quotient ring.

**Definition.**  $\dot{\mathbb{R}} := (\mathbb{R}^{\mathbb{N}}/Z)$ , as a quotient ring.

It is useful to introduce a binary relation on the set  $\mathbb{R}^{\mathbb{N}}$ , denoted by the symbol  $\doteq$ , by declaring that

$$(x_n) \doteq (x'_n) \iff x_n = x'_n \text{ for all but finitely many values of } n.$$

Equivalently,  $(x_n) \doteq (x'_n)$  if and only if there is some natural number  $N$  such that  $x_n = x'_n$  for all  $n \geq N$ . Thus, the relation  $\doteq$  is the relation of *eventual*, or *almost everywhere*, equality. One may easily check that  $\doteq$  is an equivalence relation on  $\mathbb{R}^{\mathbb{N}}$ . Thus we say that the sequences  $(x_n)$  and  $(x'_n)$  are *equivalent* if and only if  $(x_n) \doteq (x'_n)$ . Observe that  $(x_n) \doteq (x'_n)$  if and only if  $(x_n - x'_n) \in Z$ . So cosets in the quotient ring  $\mathbb{R}^{\mathbb{N}}/Z$  are precisely the same thing as the equivalence classes under  $\doteq$ . Thus,  $\dot{\mathbb{R}}$  is the set of equivalence classes of real-valued sequences under eventual equality. Intuitively, then, a fluxion may be regarded as a real sequence, with two sequences which are eventually equal representing the same fluxion.

We denote the image of a sequence  $(x_n)$  under the natural quotient map  $\mathbb{R}^{\mathbb{N}} \rightarrow \dot{\mathbb{R}}$  by  $[x_n]$ . The alternative notations  $[x_n]_{n \in \mathbb{N}}$ ,  $[x_n]_{n=1}^{\infty}$  could also be used. Addition and multiplication of fluxions is therefore defined by

$$(1.2) \quad [x_n] + [y_n] = [x_n + y_n]; \quad [x_n] \cdot [y_n] = [x_n y_n].$$

There is a natural embedding of  $\mathbb{R}$  into  $\dot{\mathbb{R}}$ , given by sending a given real number  $a$  to the corresponding constant fluxion  $[a] := [a, a, a, a, \dots]$ . Under the embedding, the additive and multiplicative identities in  $\mathbb{R}$  (0 and 1) correspond to the additive and multiplicative identities in  $\dot{\mathbb{R}}$ . It is clear that this embedding is a ring homomorphism; *i.e.*, addition and multiplication of real numbers corresponds to addition and multiplication of the corresponding fluxions. Henceforth we will regard ordinary real numbers as fluxions in this way, identifying them with their associated constant fluxion.

It is important to notice that the ring  $\dot{\mathbb{R}}$  is not a field. Indeed, the fluxion  $[(-1^n) + 1] = [0, 2, 0, 2, \dots]$  is a zero divisor, so  $\dot{\mathbb{R}}$  is not even an integral domain.

From now on we will often use single symbols such as  $x, y, z$  to denote fluxions (elements of  $\dot{\mathbb{R}}$ ). The usual ordering relation  $\leq$  on real numbers extends to a (partial) ordering of  $\dot{\mathbb{R}}$ , which will be denoted by the same symbol.

**Definition.** Given fluxions  $x = [x_n]$ ,  $y = [y_n]$ , define  $x \leq y$  if and only if  $x_n \leq y_n$  for all but finitely many  $n$ . Whenever  $x \leq y$  we say that  $x$  is *eventually* less than or equal to  $y$ . We also write  $x \geq y$  if and only if  $y \leq x$ . Whenever  $x \geq y$  we say that  $x$  is *eventually* greater than or equal to  $y$ .

Clearly, given real numbers  $a, b$  we have  $[a] \leq [b]$  if and only if  $a \leq b$ . So, when restricted to pairs of real numbers,  $\leq$  has the usual meaning. It is easily checked that  $\leq$  is a partial order on  $\dot{\mathbb{R}}$ . It is not a linear order, however. For example, if  $x = [0, 2, 0, 2, \dots] = [1 + (-1)^n]$  and  $y = [2, 0, 2, 0, \dots] = [1 - (-1)^n]$  then  $x$  and  $y$  are incomparable: neither  $x \leq y$  nor  $y \leq x$  holds.

**Definition.** Given fluxions  $x = [x_n]$ ,  $y = [y_n]$ , we write  $x < y$  if and only if  $x_n < y_n$  for all but finitely many  $n$ . Whenever  $x < y$  we say that  $x$  is *eventually* less than  $y$ . We also write  $x > y$  if and only if  $y < x$ . Whenever  $x > y$  we say that  $x$  is *eventually* greater than  $y$ .

Obviously  $x < y$  implies  $x \leq y$  and similarly  $x > y$  implies  $x \geq y$ . It should immediately be pointed out that in the realm of fluxions,  $x \leq y$  and  $x \neq y$  do not necessarily imply that  $x < y$ . For example, if  $x = [0, 2, 0, 2, \dots] = [1 + (-1)^n]$  then  $x \leq 2$  and  $x \neq 2$ , yet  $x \not< 2$  (*i.e.*,  $x$  isn't eventually smaller than 2). Similarly for  $\geq$ . Thus, a certain amount of care is required when using the ordering relations of fluxions. The main properties of  $\leq$  and  $<$  are as follows:

- (1.3)  $x \leq x$  (reflexivity of  $\leq$ );
- (1.4)  $x \leq y$  and  $y \leq x$  imply  $x = y$  (antisymmetry of  $\leq$ );
- (1.5)  $x \leq y$  and  $y \leq z$  imply  $x \leq z$  (transitivity of  $\leq$ );
- (1.6)  $x < y$  and  $y < z$  imply  $x < z$  (transitivity of  $<$ );
- (1.7)  $x \leq y \iff x - y \leq 0$ , and  $x < y \iff x - y < 0$ .

Similar statements hold when  $\leq$  and  $<$  are replaced by  $\geq$  and  $>$ .

**Definition** (Absolute value). Given a fluxion  $x = [x_n]$ , define its magnitude  $|x|$  to be the fluxion  $[|x_n|]$ .

Notice that this extends the usual notion of absolute value on real numbers. The basic properties of absolute value are essentially the same for  $\dot{\mathbb{R}}$  as for  $\mathbb{R}$ . More precisely, we have the following.

**Proposition 1.** *For any fluxions  $x, y \in \dot{\mathbb{R}}$  we have:*

- (a)  $|x - y| \geq 0$  with equality if and only if  $x = y$ ;
- (b)  $|x - y| = |y - x|$ ;
- (c)  $|x + y| \leq |x| + |y|$  (triangle inequality);
- (d) If  $y \geq 0$  then  $|x| \leq y$  if and only if  $-y \leq x \leq y$ , and similarly  $|x| < y$  if and only if  $-y < x < y$ .

The proof is immediate from the definitions.

For  $x = [x_n], y = [y_n] \in \dot{\mathbb{R}}$ , to say that  $x \neq y$  means that  $x_n \neq y_n$  for infinitely many values of  $n$ . There is a stronger form of this relation, in which inequality eventually holds everywhere, formulated as follows.

**Definition.** Let  $x = [x_n], y = [y_n] \in \dot{\mathbb{R}}$ . We say that  $x$  *avoids*  $y$  (written as  $x \# y$ ) if there is some  $N$  such that  $x_n \neq y_n$  for all  $n \geq N$ .

Obviously, if  $x \# y$  then  $x \neq y$ , but not conversely. For example, take  $x = 0$  and let  $y = [1 + (-1)^n]$ . Then  $x \neq y$  but it isn't true that  $x \# y$ .

The reader should verify that if  $x > 0$  or  $x < 0$  then  $x \# 0$ ; *i.e.*, if  $x$  is eventually positive or eventually negative, then  $x$  avoids 0. Fluxions that avoid 0 are important because they are precisely the invertible elements (units) of the ring  $\dot{\mathbb{R}}$ .

**Proposition 2.** *Let  $x, y \in \dot{\mathbb{R}}$ . Then  $x$  avoids  $y$  if and only if  $x - y$  is invertible in  $\dot{\mathbb{R}}$ . In particular,  $x \in \dot{\mathbb{R}}$  is invertible if and only if  $x$  avoids 0.*

The proof is easy.

Since  $\dot{\mathbb{R}}$  is commutative, it makes sense to define the fraction  $\frac{x}{y}$  by  $\frac{x}{y} := xy^{-1}$  whenever  $y$  is invertible. Thus division in  $\dot{\mathbb{R}}$  is well-defined, provided only that the denominator is invertible (*i.e.*, avoids 0).

To summarize: fluxions may be regarded as real sequences under the equivalence of eventual equality (similar to the way rational numbers are regarded as fractions under an equivalence). Fluxions are added, subtracted, multiplied, and divided componentwise, and this extends the usual addition, subtraction, multiplication, and division on real numbers, which are embedded in  $\dot{\mathbb{R}}$  as the constant sequences. Division makes sense only when the denominator is eventually nonzero (avoids zero). Inequalities also extend componentwise, but the notions make

sense only provided that they hold eventually. Thus the ring of fluxions merely formalizes the standard operations on sequences, familiar from basic analysis.

## 2. INFINITESIMALS

We are ready to define infinitesimals and apply them to discuss convergence. This formalizes the customary intuition that most people develop after achieving an initial understanding of convergence.

In order to foster readability, from now on we will typically write fluxions as single letters towards the end of the alphabet ( $x, y, z, \dots$ ), reserving symbols at the beginning of the alphabet ( $a, b, c, \dots$ ) to depict ordinary real numbers.

**Definition.** An *infinitesimal* is a fluxion  $x$  such that  $|x| < a$  for every positive real number  $a$ . A fluxion  $x$  is *finite* (or *bounded*) if there exists some real  $a > 0$  such that  $|x| < a$ . We write  $x \rightarrow a$  (and say that  $x$  *approaches*  $a$  or  $x$  *converges* to  $a$ ), for a real number  $a$ , if the difference  $x - a$  is infinitesimal. Finally, we say that  $x$  *diverges* if  $x$  does not converge to any real number.

When  $x \rightarrow a$  we also call the real number  $a$  the *limit* of  $x$ , so we have already defined limits of sequences. One may prefer to employ one of the more cumbersome notations

$$\lim x = a, \lim x_n = a, \text{ or } \lim_{n \rightarrow \infty} x_n = a$$

as an alternative notation for  $x \rightarrow a$ .

**Examples.** As an example of a nonzero infinitesimal, consider  $h = [1/n] \in \dot{\mathbb{R}}$ . Notice that  $x$  is infinitesimal if and only if  $|x|$  is infinitesimal if and only if  $-x$  is infinitesimal. Clearly 0 is infinitesimal; moreover, 0 is the only real infinitesimal. It is clear that any real number or infinitesimal is a finite fluxion. Furthermore, any convergent fluxion is finite, because  $x - a = h$  with  $h$  infinitesimal implies that  $|x| = |a + h| \leq |a| + |h| < |a| + 1$ .

Algebraic interactions among finite fluxions are encapsulated in the following result.

**Proposition 3.** *The set  $\mathbb{F}$  of finite fluxions is a subring of  $\dot{\mathbb{R}}$ , and the set  $\mathbb{O}$  of infinitesimals is an ideal in  $\mathbb{F}$ . The set  $\mathbb{M}$  of convergent fluxions is a subring of  $\mathbb{F}$ , and  $\mathbb{M} = \mathbb{R} \oplus \mathbb{O}$ .*

*Proof.* Let  $x, y \in \mathbb{F}$ . Then there are positive reals  $a, b$  such that  $|x| < a$  and  $|y| < b$ . Hence  $|x - y| \leq |x| + |y| < a + b$ , so  $x - y \in \mathbb{F}$ . Since  $|xy| = |x||y| < ab$ , it also follows that  $xy \in \mathbb{F}$ . So  $\mathbb{F}$  is a subring of  $\mathbb{R}$ .

Now let  $h, i \in \mathbb{O}$  and  $x \in \mathbb{F}$ . So  $|x| < a$  for some positive real  $a$ , while  $|h| < \varepsilon$  and  $|i| < \varepsilon$  for every positive real  $\varepsilon$ . Then

$$|h - i| \leq |h| + |i| < \varepsilon + \varepsilon = 2\varepsilon,$$

for every positive real  $\varepsilon$ , so  $h - i \in \mathbb{O}$ . Moreover,  $|xh| = |x||h| < a\varepsilon$ , for every positive real  $\varepsilon$ , so  $xh \in \mathbb{O}$ .

Finally,  $\mathbb{R} \cap \mathbb{O} = \{0\}$  since 0 is the only real infinitesimal. Any convergent  $x$  is finite, and can be written in the form  $x = a + h$  for some real  $a$ , and some infinitesimal  $h$ .  $\square$

The proposition says, in particular, that *sums, differences, and products of finite fluxions are finite*, and similarly *sums, differences, and products of infinitesimals are infinitesimal*. Furthermore, *a product of a finite fluxion and an infinitesimal is infinitesimal*. Finally, *every convergent fluxion is uniquely expressible as the sum of a real number and an infinitesimal*.

Notice that it follows immediately from the proposition that limits are unique: if  $x \rightarrow a$  and  $x \rightarrow a'$  for  $a, a' \in \mathbb{R}$  then  $a = a'$ . For  $x - a = h$  and  $x - a' = h'$  where  $h, h'$  are infinitesimal, so the difference  $a - a' = h' - h$  is infinitesimal, but since both  $a, a'$  are real this forces  $a = a'$ , as desired.

It does not seem possible to obtain a meaningful geometric model of the entire ring  $\mathbb{R}$  of fluxions. However, if we restrict our attention to the subring  $\mathbb{M}$  of convergent fluxions, then we could define the *monad* of a real number  $a$  to be  $\text{monad}(a) := \{x \in \mathbb{R} : x \rightarrow a\}$ , and visualize this set as an infinitesimal “cloud” surrounding  $a$ , intersecting the real line in just the point  $a$ . While  $\mathbb{M}$  itself is not linearly ordered, the monads are:  $\mathbb{M}$  is the disjoint union of its monads, and could be roughly visualized as a “cloudy” line in which each point on the usual real line has been expanded to its corresponding monad.

Next we consider infinite limits.

**Definition.** Let  $x$  be a fluxion. Write  $x \rightarrow \infty$  ( $x$  approaches infinity) if  $x > a$  for every real  $a$ . Similarly, write  $x \rightarrow -\infty$  ( $x$  approaches negative infinity) if  $x < a$  for every real  $a$ .

For example,  $[n] \rightarrow \infty$  and  $[-n] \rightarrow -\infty$ . We should reiterate that if  $x \in \mathbb{R}$  approaches  $\pm\infty$  then  $x$  diverges. Moreover, there are many examples (e.g.,  $[(-1)^n n]$ ,  $[(-1)^n]$ ) of fluxions which diverge, yet neither approach infinity nor approach minus infinity.

The following is a useful characterization of fluxions which diverge to  $\pm\infty$ . Let us introduce the notations  $\mathbb{O}^+ := \{x \in \mathbb{O} : x > 0\}$ ,  $\mathbb{O}^- := \{x \in \mathbb{O} : x < 0\}$ . These are the sets of eventually positive or eventually negative infinitesimals.

**Proposition 4.** *Let  $x$  be a fluxion. Then  $x \rightarrow \infty \iff 1/x \in \mathbb{O}^+$  and  $x \rightarrow -\infty \iff 1/x \in \mathbb{O}^-$ .*

*Proof.* Suppose  $x \rightarrow \infty$ . Then  $x > a$  for any real number  $a$ , so  $x > 0$ . Thus  $x$  is invertible,  $1/x > 0$ , and since  $1/x < 1/a$  for any real  $a > 0$ , it follows that  $1/x$  is smaller than any positive real number, hence is infinitesimal. Conversely, if  $1/x > 0$  is infinitesimal then  $0 < 1/x < a$  for every positive real number  $a$ , so  $x > 1/a$  for every positive real  $a$ , and it follows that  $x > b$  for every  $b \in \mathbb{R}$ , so  $x \rightarrow \infty$ .

The negative case is similar. □

In particular, if  $x \rightarrow \pm\infty$  then  $x$  is invertible. The following result gives conditions under which the reciprocal of a fluxion exists and is finite. This is often useful in calculations.

**Proposition 5.** *Let  $x$  be a fluxion.*

(a) *Suppose that  $x$  is bounded away from 0, meaning that there is some real number  $b$  such that  $0 < b < |x|$ . Then  $x$  avoids 0, so  $x^{-1} = 1/x$  exists in  $\mathbb{R}$ , and moreover  $1/x$  is finite.*

(b) *If  $x \rightarrow a$  for  $a \neq 0$  real, then  $|a| - \frac{1}{n} < |x| < |a| + \frac{1}{n}$  for any natural number  $n$ . Hence  $1/x$  exists and is finite.*

*Proof.* (a) Write  $x = [x_n]$ . There must be some  $N$  for which  $b < |x_n|$  for all  $n \geq N$ , so in particular (by replacing the first  $N - 1$  terms by something nonzero if necessary) we may assume that  $x_n \neq 0$  for all  $n$ , and thus  $1/x = [1/x_n]$  makes sense. Clearly  $b < |x|$  implies  $|1/x| < 1/b$ , so  $1/x$  is finite.

(b) Put  $h = x - a$ . Then  $h$  is infinitesimal, so  $|h| < \frac{1}{n}$  for any  $n \in \mathbb{N}$ . Then  $|x| = |a + h| \leq |a| + |h| < |a| + \frac{1}{n}$ , for any  $n$ . Similarly,  $|x| = |a - (-h)| \geq ||a| - |h|| > |a| - \frac{1}{n}$ , for any  $n$ . In particular, by taking  $n$  so large that  $\frac{1}{n} < |a|$  (such an  $n$  exists by the archimedean property of  $\mathbb{R}$ ) we see that  $|x| > |a| - \frac{1}{n}$ , i.e.,  $x$  is bounded away from 0. □

The following summarizes how convergence interacts with algebraic operations.



**Proposition 6** (Algebra of Convergence). *Suppose that  $x \rightarrow a$  and  $y \rightarrow b$  where  $a, b$  are real. Let  $c \in \mathbb{R}$ . Then  $x + y \rightarrow a + b$ ,  $x - y \rightarrow a - b$ ,  $cy \rightarrow cb$ ,  $xy \rightarrow ab$ , and  $x/y \rightarrow a/b$  (provided  $b \neq 0$ ).*

*Proof.* Let  $x - a = h$ ,  $y - b = k$  where  $h, k$  are infinitesimals. Then  $(x + y) - (a + b) = h + k$ ,  $(x - y) - (a - b) = h - k$ ,  $cy - cb = ck$ ,  $xy - ab = xy - ay + ay - ab = (x - a)y + a(y - b) = hy + ak$ , and

$$\frac{x}{y} - \frac{a}{b} = \frac{bx - ay}{by} = \frac{b(x - a) + a(b - y)}{by} = \frac{bh - ak}{by}$$

are all infinitesimal. (Proposition 5 shows that  $y$  is invertible in  $\mathbb{R}$  if  $b \neq 0$ .)  $\square$

Next we consider how convergence interacts with inequalities.

**Proposition 7** (Convergence and Inequalities). *Let  $a, b$  be real numbers, and  $x, y, z$  fluxions.*

- (a) *If  $a + h \geq 0$  for some infinitesimal  $h$ , then  $a \geq 0$ .*
- (b) *If  $x \leq y$ ,  $x \rightarrow a$  and  $y \rightarrow b$  then  $a \leq b$ .*
- (c) *If  $x \leq y \leq z$ ,  $x \rightarrow a$  and  $z \rightarrow a$  then  $y \rightarrow a$ .*

*Proof.* (a) Assume  $a < 0$ . Then  $-a > 0$  and  $h \geq -a$ . This contradicts the assumption that  $h$  is infinitesimal.

(b) Write  $x = a + h$ ,  $y = b + k$  where  $h, k$  are infinitesimals. Then  $x - y = (a - b) + (h - k)$ , so  $(a - b) + (h - k) \geq 0$ . By part (a), it follows that  $a - b \geq 0$ .

(c) Since  $x - a \leq y - a \leq z - a$  it follows that  $y - a$  is infinitesimal. (To be precise,  $x - a \leq y - a \leq z - a$  implies that  $|y_n - a| \leq \max(|x_n - a|, |z_n - a|)$  for all sufficiently large  $n$ , which shows that  $y - a$  is infinitesimal.) Thus  $y \rightarrow a$ .  $\square$

The reader can easily verify that part (b) of the preceding result admits the following generalization: if  $x \leq y$ ,  $x \rightarrow a$  and  $y \rightarrow b$  for  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  then  $a \leq b$ . For this interpretation, one has to stipulate that  $-\infty < a < \infty$  for all  $a \in \mathbb{R}$ . It should also be noted that part (b) fails for strict inequalities: it is possible to find examples where  $x < y$ ,  $x \rightarrow a$ , and  $y \rightarrow b$  yet  $a \not\leq b$ .

The following result is a fundamental property of convergence. We say that  $x \in \mathbb{R}$  is *monotonic* if  $x = [x_n]$  is eventually either monotonically increasing ( $x_n \leq x_{n+1}$  for all sufficiently large  $n$ ) or monotonically decreasing ( $x_n \geq x_{n+1}$  for all sufficiently large  $n$ ).

**Theorem** (Monotone Convergence Theorem). *A monotonic fluxion is finite if and only if it converges.*

*Proof.* We argue only the increasing case, as the decreasing case is similar. Assume  $x$  is monotonically increasing, *i.e.*,  $x_n \leq x_{n+1}$  for all  $n \geq N$ . If  $x$  converges then  $x$  is finite. Conversely, suppose that  $x$  is finite. Let  $E = \{x_n : n \geq N\}$ . The set  $E$  is a bounded set of real numbers, so its supremum  $a = \sup E$  exists in  $\mathbb{R}$ . Then it is easy to see from the definition of supremum that  $x - a$  is infinitesimal, and hence  $x \rightarrow a$ .  $\square$

It should be clear that if a fluxion  $x$  is monotonically increasing yet not finite, then  $x \rightarrow \infty$ , and similarly, if  $x$  is monotonically decreasing yet not finite, then  $x \rightarrow -\infty$ .

**Definition.** If  $(x_{n_k})_{k \in \mathbb{N}}$  is a subsequence of a sequence  $(x_n)_{n \in \mathbb{N}}$  defining a fluxion  $x = [x_n]$ , put  $y = [x_{n_k}]$  and say that the fluxion  $y$  is a *subfluxion* of  $x$  (written as  $y \subset x$ ).

The Bolzano–Weierstrass theorem is another fundamental result in analysis. The proof given below is taken from a Wikipedia article.

**Theorem** (Bolzano–Weierstrass Theorem). *Any finite fluxion must have a convergent subfluxion.*

*Proof.* Let  $(x_n)$  be a sequence such that the corresponding fluxion  $x = [x_n]$  is finite. We claim that there must be a monotonic subsequence of  $(x_n)$ ; the corresponding fluxion must converge by the monotone convergence theorem.

To prove the claim, let us call a positive integer  $n$  a *peak* of the sequence  $(x_n)$  if  $x_n > x_m$  for all  $m > n$ ; *i.e.*, if  $x_n$  is greater than every subsequent term in the sequence. Suppose first that the sequence has infinitely many peaks,  $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ . Then the subsequence  $(x_{n_k})$  corresponding to peaks is monotonically decreasing, and we are done. So suppose now that there are only finitely many peaks, let  $N$  be the last peak and put  $n_1 = N + 1$ . Then  $n_1$  is not a peak, since  $n_1 > N$ , which implies the existence of an  $n_2 > n_1$  with  $x_{n_2} \geq x_{n_1}$ . Again, any  $n_2 > N$  is not a peak, hence there is some  $n_3 > n_2$  with  $x_{n_3} \geq x_{n_2}$ . Repeating this process leads to an infinite monotonically increasing subsequence  $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots$ . Finally, if there are no peaks at all, then for every element of the sequence, there must be a subsequent larger element, which, in turn has a subsequent larger element and so on, and these constitute a monotonic subsequence. This proves the claim, and thus the result.  $\square$

We now discuss limits inferior and superior. For this we need to work within the extended real-number system  $\mathbb{R} \cup \{\pm\infty\}$ .

**Definition.** Suppose  $x$  is a fluxion. Let  $E \subset \mathbb{R} \cup \{\pm\infty\}$  be the collection of all extended real numbers  $a \in \mathbb{R} \cup \{\pm\infty\}$  for which there is some subfluxion  $y \subset x$  such that  $y \rightarrow a$ . Define

$$\limsup x = \sup E, \quad \liminf x = \inf E.$$

The extended real numbers  $\limsup x$  and  $\liminf x$  are respectively called the *limit superior* and *limit inferior* of the fluxion  $x$ .

It should be clear to the reader that if  $x \rightarrow a$  then also  $y \rightarrow a$ , for any  $y \subset x$ . This is true for any  $a \in \mathbb{R} \cup \{\pm\infty\}$ . This leads to the following characterization of convergence (and divergence to  $\pm\infty$ ) in terms of the limit superior and limit inferior.

**Proposition 8.** *Let  $x$  be a fluxion. Then  $x \rightarrow a \in \mathbb{R} \cup \{\pm\infty\}$  if and only if  $\limsup x = a = \liminf x$ .*

*Proof.* Suppose  $x \rightarrow a$ . Then  $y \rightarrow a$  for any  $y \subset x$ , and thus  $E = \{a \in \mathbb{R} \cup \{\pm\infty\} : y \rightarrow a, y \subset x\}$  is a singleton  $\{a\}$ , whence  $\sup E = \inf E = a$ . On the other hand, if  $\limsup x = a = \liminf x$  then the set  $E$  defined above must be a singleton  $\{a\}$ , and so every subfluxion of  $x$  must approach  $a$ . In particular,  $x \rightarrow a$ .  $\square$

Here is an elementary application of the last result, which illustrates the utility of these concepts.

**Proposition 9.** *Let  $x \geq 0$  be a fluxion and suppose that  $x \rightarrow a$ , where  $a$  is real. Then  $\sqrt[n]{x} \rightarrow \sqrt[n]{a}$ , for any  $n \in \mathbb{N}$ .*

*Proof.* (Based on a suggestion by R. Jensen.) First we claim that if a fluxion  $y \geq 0$  diverges then  $y^n$  diverges. (The claim is false without the assumption that  $y \geq 0$ .) If  $y$  isn't finite then neither is  $y^n$ , so the claim is proved in that case. On the other hand, if  $y$  is finite then by the preceding result we must have  $\limsup y \neq \liminf y$ , so there exist subfluxions  $u, v \subset y$  with  $u \rightarrow b, v \rightarrow c$  for real  $b \neq c$ . Then  $u^n, v^n \subset y^n$  and  $u^n \rightarrow b^n, v^n \rightarrow c^n$ , and  $b^n \neq c^n$  so  $y^n$  diverges in this case, too.

Now put  $y = \sqrt[n]{x}$ . The contrapositive of the claim proved in the preceding paragraph shows that  $y$  converges. Let  $d$  be the limit, so  $y \rightarrow d$ . Then by Proposition 6 it follows that  $y^n \rightarrow d^n$ , so by uniqueness of limits  $d^n = a$  and thus  $d = \sqrt[n]{a}$ , as desired.  $\square$

Convergence of an infinite series  $\sum_{n=1}^{\infty} a_n$  may be defined as usual, in terms of convergence of the fluxion  $s = [s_n] = [a_1 + \cdots + a_n]$ .

### 3. CONTINUITY AND DIFFERENTIABILITY

We now consider continuity. For this, we need to extend real-valued functions of a real variable to assume values at fluxions.

If  $f$  is a real-valued function of a real variable, let  $D \subset \mathbb{R}$  be its domain. So  $f$  maps  $D$  into  $\mathbb{R}$ . Let  $\dot{D}$  be the set of all fluxions  $x = [x_n]$  which are eventually in  $D$ , meaning that  $x_n \in D$  for all but finitely many  $n$ . Then  $f$  extends to a function, also denoted by  $f$ , from  $\dot{D}$  into  $\dot{\mathbb{R}}$ , by defining  $f(x)$  to be  $[f(x_n)]$  if  $x = [x_n] \in \dot{D}$ .

**Definition.** Let  $f: D \rightarrow \mathbb{R}$  be a real-valued function of a real variable, and let  $a$  be a point of its domain  $D$ . We say that  $f$  is *continuous at*  $a$  if  $f(x) \rightarrow f(a)$ , for every fluxion  $x \in \dot{D}$  for which  $x \rightarrow a$ . One also says that  $f$  is *continuous on*  $D$  if  $f$  is continuous at  $a$  for every  $a \in D$ .

Note that if  $a$  is an isolated point of  $D$ , meaning that the only  $x \in \dot{D}$  converging to  $a$  is  $a$  itself, then any  $f: D \rightarrow \mathbb{R}$  is continuous at  $a$ . The definition also includes the notions of left or right continuity, in case the point  $a$  is a left or right endpoint of  $D$ .

**Proposition 10** (Algebra of Continuity). *Let  $f, g$  be real-valued functions of a real variable, both continuous at some real point  $a$ . Then  $f + g$ ,  $f - g$ , and  $fg$  are continuous at  $a$ . Moreover,  $f/g$  is continuous at  $a$  provided that  $g(a) \neq 0$ .*

*Proof.* Combine the definition with Proposition 6. □

As an application, notice that Proposition 9 says that the real-valued function  $x \mapsto x^{1/n}$  is continuous on the interval  $[0, \infty)$ . It then follows from the proposition that the real-valued function  $x \mapsto x^{m/n}$  is continuous on the interval  $[0, \infty)$ ; that is, taking rational powers is a continuous operation.

**Theorem** (Intermediate Value Theorem). *Let  $f$  be a real-valued function defined and continuous (at least) on the interval  $[a, b]$ , such that  $f(a) \neq f(b)$ . If  $i$  is any real number strictly between  $f(a)$  and  $f(b)$ , then there exists some real number  $c$  strictly between  $a$  and  $b$  such that  $f(c) = i$ .*

*Proof.* A standard argument, based on the bisection method, easily translates into the language of fluxions. First we observe that by replacing  $f$  by  $f - i$  we are reduced to the case where the intermediate value  $i$  is zero, and  $f(a)$  and  $f(b)$  have different signs (i.e.,  $f(a)f(b) < 0$ ). Then we put  $x_0 = a$ ,  $y_0 = b$ . Computing the midpoint  $m_0 = \frac{1}{2}(x_0 + y_0)$  of the interval  $[x_0, y_0]$ , we have three cases:

- (1) If  $f(m_0) = 0$  then we are done (put  $c = m_0$ ).
- (2) Otherwise, if  $f(m_0)$  has the same sign as  $f(x_0)$ , put  $x_1 = m_0$  and  $y_1 = y_0$ .
- (3) Otherwise, put  $x_1 = x_0$  and  $y_1 = m_0$ .

Assuming we didn't already find  $c$ , this produces a new interval  $[x_1, y_1]$ , half the size of the original, satisfying  $f(x_1)f(y_1) < 0$ . So the process can be repeated, to produce a new interval  $[x_2, y_2]$ , and so on. Either this process terminates after finitely many steps (by finding some  $m_n = \frac{1}{2}(x_n + y_n)$  such that  $f(m_n) = 0$ ) or we obtain two infinite sequences  $(x_n)$ ,  $(y_n)$  defining fluxions  $x := [x_n]$ ,  $y := [y_n]$ . Now in the latter case both  $x$  and  $y$  are monotonic, and hence convergent by the monotone convergence theorem. Furthermore,  $y - x \rightarrow 0$  is infinitesimal since  $y_n - x_n = \frac{1}{2^n}(b - a)$  for all  $n$ , so  $x$  and  $y$  have the same limit  $c$ . Finally, by construction all the  $f(x_n)$  have the same sign as  $f(x_0) = f(a)$ , and all the  $f(y_n)$  have the same sign as  $f(y_0) = f(b)$ . It follows that either  $f(x) > 0$  and  $f(y) < 0$ , or else  $f(x) < 0$  and  $f(y) > 0$ . In either case, both  $f(x)$  and  $f(y)$  converge to  $f(c)$  by continuity of  $f$ , so by Proposition 7 it follows that  $0 \leq f(c) \leq 0$ , and  $f(c) = 0$ .  $\square$

Next we consider uniform continuity. In order to formulate it, we require the following concept.

**Definition.** Two fluxions  $x, y$  are *infinitely close* (written as  $x \approx y$ ) if their difference  $x - y$  is infinitesimal.

It is easily checked that infinite closeness is an equivalence relation on the set  $\mathbb{R}$  of fluxions. Notice that if  $a$  is real then  $x \approx a$  if and only if  $x \rightarrow a$ . It is easy to construct examples of divergent fluxions  $x, y$  with  $x \approx y$ ; for instance  $x = [n]$  and  $y = [n + 1/n]$ .

**Definition.** Let  $f$  be a real-valued function of a real variable, and let  $E$  be a subset of its domain. We say that  $f$  is *uniformly continuous* on the set  $E$  if  $f(x) \approx f(y)$  whenever  $x \approx y$ , for any fluxions  $x, y \in E$ .

It is clear that if  $f$  is uniformly continuous on  $E$  then  $f$  is continuous on  $E$  (i.e., continuous at every point of  $E$ ). The converse fails. For example, the function  $f(x) = x^2$  is continuous on  $\mathbb{R}$  but not uniformly continuous on  $\mathbb{R}$ , because  $x = [n]$  and  $y = [n + \frac{1}{n}]$  are infinitely close inputs producing outputs

$$f(x) = [n^2], \quad f(y) = [n^2 + 2 + 1/n^2]$$

that are not infinitely close.

Let us now consider differentiability. Some slight care is required in order to ensure that the difference quotient is defined for all fluxions under consideration.

**Definition.** Let  $f: D \rightarrow \mathbb{R}$  be a real-valued function of a real variable. Let  $a$  be a point of  $D$  with the property that there is some fluxion  $x \in \dot{D}$  with  $x \rightarrow a$  but such that  $x$  avoids  $a$ . (Such  $a$  are called accumulation points of  $D$ .) Then  $f$  is said to be *differentiable* at  $a$  provided that  $\frac{f(x)-f(a)}{x-a}$  converges to a real number  $f'(a)$  for every fluxion  $x \in \dot{D}$  with  $x$  avoiding  $a$  and  $x \rightarrow a$ . The real number  $f'(a)$  is called the derivative of  $f$  at  $a$ .

One could derive the standard rules of differentiation at this point.

**Proposition 11.** *Differentiability implies continuity.*

*Proof.* Put  $h = x - a$ , where  $x \in \dot{D}$  and  $x$  avoids  $a$ . Then  $h$  is an invertible infinitesimal and  $\frac{f(x)-f(a)}{h} \rightarrow f'(a)$ . Since  $h \rightarrow 0$  it follows that  $h \cdot \frac{f(x)-f(a)}{h} \rightarrow 0 \cdot f'(a)$ . In other words,  $f(x) - f(a) \rightarrow 0$ , so  $f(x) \rightarrow f(a)$ . If  $x \in \dot{D}$  converges to  $a$  but doesn't avoid  $a$ , then it also follows that  $f(x) \rightarrow f(a)$ , so the proof is complete.  $\square$

Differentials may be treated as usual, by introducing  $dx$  as another independent variable, and if  $y = f(x)$  is a function, defining  $dy$  by the rule  $dy = f'(x)dx$ . If  $dx$  is allowed to take on infinitesimal values, then  $dy$  becomes infinitesimal as well, for all values of  $x$  for which  $f$  is differentiable.

Limits of functions can also be defined using infinitesimal language. The notion of invertibility in the ring  $\mathbb{R}$  is a crucial ingredient of the definition. For the sake of economy, it is convenient to work within the extended real number system  $\mathbb{R} \cup \{\infty, -\infty\}$ . This allows the various cases of infinite limits as well as finite ones to be treated with a single definition.

**Definition.** Let  $f: D \rightarrow \mathbb{R}$  be a real-valued function of a real variable, and suppose that there is some fluxion  $x \in \dot{D}$  with  $x$  avoiding  $a$ , such that  $x \rightarrow a$  for some extended real number  $a$ . We say that  $f(x)$  approaches an extended real number  $L$  as  $x$  approaches  $a$ , and write

$$\lim_{x \rightarrow a} f(x) = L$$

if  $f(x) \rightarrow L$  for every  $x \rightarrow a$  such that  $x$  avoids  $a$ .

In other words, the equality  $\lim_{x \rightarrow a} f(x) = L$  means that  $f(x)$  approaches  $L$  for every  $x \in \dot{D}$  that approaches and avoids  $a$ , and that

there must be at least one such  $x \in \dot{D}$ . The existence and value of  $f(a)$ , if any, is irrelevant for the existence and value of the limit.

We note also that one-sided limits can be obtained as a special case of the definition, by merely restricting the domain appropriately. The details are left to the reader.

At this point one may prove the following standard property of continuity, sometimes taken as its definition.

**Proposition 12.** *Let  $f: D \rightarrow \mathbb{R}$  be a real-valued function of a real variable, and suppose  $a$  is a point of  $D$ . Then  $f$  is continuous at  $a$  if and only if  $\lim_{x \rightarrow a} f(x) = f(a)$ .*

*Proof.* From the definitions it is clear that if  $f$  is continuous at  $a$  then  $\lim_{x \rightarrow a} f(x) = f(a)$ . Conversely, suppose that  $\lim_{x \rightarrow a} f(x) = f(a)$ . Then  $f(x) \rightarrow f(a)$  for all  $x \rightarrow a$  such that  $x$  avoids  $a$ . We need to show that  $f(x) \rightarrow f(a)$  for all  $x \rightarrow a$ , even if  $x$  does not avoid  $a$ . So assume that  $x$  does not avoid  $a$  and  $x \rightarrow a$ . This means that if we write  $x = [x_n]$  then for any  $N \in \mathbb{N}$  there are infinitely many values of  $n \geq N$  such that  $x_n = a$ . It is still true that  $f(x) \rightarrow f(a)$ , so the proof is complete.  $\square$

#### 4. TOPOLOGY

The language of infinitesimals is appropriate for discussing the topology of the real line. Indeed, we have already encountered isolated points and accumulation points. Here are infinitesimal definitions of the standard topological notions.

**Definition.** Let  $E$  be a subset of  $\mathbb{R}$ . The set  $E$  is *closed* if every convergent fluxion in  $\dot{E}$  converges to a point of  $E$  (i.e.,  $x \in \dot{E}$  and  $x \rightarrow a$  for  $a$  real implies  $a \in E$ ). A real point  $a$  is said to be an *accumulation point* of  $E$  if there is some  $x \in \dot{E}$  such that  $x$  avoids  $a$  and  $x \rightarrow a$ . A point  $a \in E$  is said to be an *isolated* point of  $E$  if the only  $x \in \dot{E}$  converging to  $a$  is  $x = a$ . Finally, we say that  $E$  is *open* if for every  $a \in E$ ,  $\dot{E}$  contains every fluxion  $x \in \dot{\mathbb{R}}$  which converges to  $a$  (i.e.,  $x \in \dot{\mathbb{R}}$  and  $x \rightarrow a \in E$  implies  $x \in \dot{E}$ ).

Clearly,  $E$  is closed if and only if  $E$  contains all of its limits. The limit of a fluxion in  $\dot{E}$  either belongs to  $E$  or is an accumulation point of  $E$ , so  $E$  is closed if and only if  $E$  contains all of its accumulation points. A finite set has no accumulation points since all its points are isolated, so finite sets are closed.

The notion of accumulation point already arose in the discussion of limits of functions. Looking back at the definition, we see that in

order for  $\lim_{x \rightarrow a} f(x)$  to exist in the extended real number system, it is necessary that  $a$  be an accumulation point for the domain of  $f$ .

In order to prove some of the standard topological results about  $\mathbb{R}$ , it is useful to have the following lemma available.

**Lemma.** *Let  $E, F$  be subsets of  $\mathbb{R}$ , and  $\{E_\alpha\}$  a family of subsets of  $\mathbb{R}$ . Then*

- (a)  $E \subset F$  implies  $\dot{E} \subset \dot{F}$ ;
- (b)  $\bigcup (\dot{E}_\alpha) \subset \dot{U}$ , where  $U := (\bigcup E_\alpha)$ ;
- (c)  $\bigcap (\dot{E}_\alpha) = \dot{V}$ , where  $V := (\bigcap E_\alpha)$ ;
- (d)  $E \cap F = \emptyset$  implies  $\dot{E} \cap \dot{F} = \emptyset$ .
- (e) If  $F = \mathbb{R} - E$  is the complement of  $E$ , then  $\dot{F} \subset \mathbb{R} - \dot{E}$ .

*Proof.* Part (a) is obvious. Part (b) simply says that a fluxion whose range lies in some  $E_\alpha$  lies in the union of all  $E_\alpha$ , so that is also clear.

To get (c), notice that  $(\bigcap E_\alpha) \subset E_\alpha$  implies by part (a) that  $\dot{V} \subset \dot{E}_\alpha$ . Since this holds for any  $\alpha$ , we have  $\dot{V} \subset \bigcap (\dot{E}_\alpha)$ . For the opposite inclusion, observe that if  $x \in \bigcap (\dot{E}_\alpha)$  then for any  $\alpha$ ,  $x \in \dot{E}_\alpha$ , so  $x \in \dot{V}$ , as desired.

Part (d) is obvious (it also follows from part (c)), and part (e) follows immediately from part (d).  $\square$

The lemma is used in the proofs of the next two results.

**Proposition 13.** *A set  $E$  of reals is open if and only if its complement  $\mathbb{R} - E$  is closed.*

*Proof.* Suppose that  $E$  is open. Let  $F$  be the complement of  $E$  and suppose  $x \in \dot{F} = \mathbb{R} - \dot{E}$ . Assume that  $x \rightarrow a$  where  $a$  is real. If  $a \in E$  then  $x \in \dot{E}$ , a contradiction. So  $a \in F$  and thus  $F$  is closed.

On the other hand, if  $F = \mathbb{R} - E$  is closed, consider any real  $a \in E$  and any fluxion  $x$  with  $x \rightarrow a$ . We must show that  $x \in \dot{E}$ . If not, then any sequence  $(x_n)$  defining  $x$  contains infinitely many points of  $F$ , and thus there would be a subfluxion  $y$  of  $x$  with  $y \in \dot{F}$ . Since  $y \rightarrow a$  and  $F$  is closed, this implies that  $a \in F$ , a contradiction. Thus  $x \in \dot{E}$ , as required.  $\square$

**Proposition 14.** *The union of any collection of open sets in  $\mathbb{R}$  is again open. A finite intersection of open sets in  $\mathbb{R}$  is open. The empty set and  $\mathbb{R}$  itself are both open (and closed). Thus the collection of open subsets of  $\mathbb{R}$  forms a topology on  $\mathbb{R}$ .*



*Proof.* Let  $\{G_\alpha\}$  be an arbitrary collection of open sets, and put  $G = \bigcup_\alpha G_\alpha$ . If  $a \in G$  then  $a \in G_\alpha$  for some  $\alpha$ , and thus every fluxion  $x$  such that  $x \rightarrow a$  belongs to  $\dot{G}_\alpha \subset \dot{G}$ , so  $G$  is open.

Now suppose we have a finite collection  $\{G_1, \dots, G_k\}$  of open sets, and put  $H = G_1 \cap G_2 \cap \dots \cap G_k$ . If  $a \in H$  then  $a \in G_j$  for each  $j = 1, \dots, k$ , so any fluxion  $x$  such that  $x \rightarrow a$  has the property that  $x \in \dot{G}_j$ , for  $j = 1, \dots, k$ . But then  $x \in \dot{G}_1 \cap \dot{G}_2 \cap \dots \cap \dot{G}_k = \dot{H}$ , and thus  $H$  is open in  $\mathbb{R}$ .  $\square$

Compactness may also be defined via infinitesimals. (This is based on the nontrivial fact that for metric spaces, compactness and sequential compactness are equivalent.)

**Definition.** Let  $K$  be a subset of  $\mathbb{R}$ . We say that  $K$  is *compact* if every fluxion  $x \in \dot{K}$  has a convergent subfluxion  $u \subset x$ , converging to a point of  $K$ .

Clearly, any finite set  $K = \{a_1, a_2, \dots, a_k\}$  of real numbers is compact, because any fluxion  $x \in \dot{K}$  must have a constant subfluxion.

Here are some simple consequences of the definition of compactness.

**Proposition 15.** *Let  $K \subset \mathbb{R}$  be compact.*

- (a)  *$K$  is closed.*
- (b) *Any closed subset  $F$  of  $K$  is compact.*
- (c) *Any infinite subset  $E$  of  $K$  must have an accumulation point in  $K$ .*

*Proof.* (a) If  $x \in \dot{K}$  converges, then any subfluxion of  $x$  also converges, and converges to the same limit, which must be in  $K$  by compactness of  $K$ . So  $K$  contains all its limits, and thus is closed.

(b) Suppose that  $x \in \dot{F}$ . Since  $\dot{F} \subset \dot{K}$ , by compactness of  $K$  there is a subfluxion  $u \subset x$  such that  $u \rightarrow a$  for some  $a \in K$ . But  $u \in \dot{F}$  and  $F$  is closed, so in fact  $a \in F$ , as required.

(c) Pick some  $x \in \dot{E}$  such that  $x$  has no constant subfluxion. This is possible because  $E$  is infinite. Since  $\dot{E} \subset \dot{K}$ , by compactness of  $K$  there must be some subfluxion  $u \subset x$  such that  $u \rightarrow a \in K$ . Since  $x$  has no constant subfluxion,  $u$  avoids  $a$ , and thus  $a \in K$  is an accumulation point of  $E$ . (Note that  $u \in \dot{E}$  since  $x \in \dot{E}$ .)  $\square$

It follows immediately from the Bolzano–Weierstrass theorem that any closed interval  $I = [a, b]$  is compact in  $\mathbb{R}$ . (Because  $I$  is closed, the limit of a convergent subfluxion of any  $x \in \dot{I}$  must lie in  $I$ .)

**Theorem** (Heine–Borel Theorem). *Let  $E$  be a subset of  $\mathbb{R}$ . Then  $E$  is compact if and only if  $E$  is closed and bounded.*

*Proof.* Assume  $E$  is compact. Then by Proposition 15(a),  $E$  is closed. To see that  $E$  is bounded, argue by contradiction. Assuming the contrary, we can find a fluxion  $x = [x_n]$  in  $\dot{E}$  such that  $|x_n| > n$  for each natural number  $n$ . The fluxion  $x$  is not finite. No subfluxion of  $x$  can converge to a point of  $E$ , since convergent fluxions are finite. This violates the compactness of  $E$ .

For the converse, assume that  $E$  is closed and bounded. Since  $E$  is bounded, we can find some interval  $I = [-r, r]$  containing  $E$ . Since  $I$  is compact, by Proposition 15(b) it follows that  $E$  is compact.  $\square$

As an application of the last result, we can prove that every continuous real-valued function of a real variable achieves a maximum and minimum on any closed interval. For this we first need to observe that *the continuous image of a compact set must be compact*. For if  $f$  is continuous and  $E$  a compact subset of its domain, then put  $F := f(E)$ . If  $y = f(x) \in \dot{F}$  for some  $x \in \dot{E}$ , then by compactness of  $E$  there is a subfluxion  $u \subset x$  with  $u \rightarrow a$  where  $a$  is real and  $a \in E$ . By continuity,  $f(u) \rightarrow f(a) \in F$ , and clearly  $f(u) \subset f(x)$ .

**Theorem** (Extreme Values Theorem). *A continuous real-valued function  $f: D \rightarrow \mathbb{R}$  of a real variable attains a maximum and minimum on any closed interval  $I = [a, b]$  contained in its domain.*

*Proof.* Put  $K = f(I)$ . Since  $K$  is compact, it is closed and bounded by the Heine–Borel theorem. Boundedness implies the existence of the real numbers  $m := \inf K$  and  $M := \sup K$ . We need to show that both  $m$  and  $M$  belong to  $K$ . But surely there is a sequence of points in  $K$  approaching  $m$ . By compactness, this forces  $m \in K$ . Similarly for  $M$ .  $\square$

From this one can derive the mean value theorem and its various consequences (*e.g.*, Taylor’s theorem).

One could go on to treat the topology of metric spaces similarly, in particular extending the results of this section to subsets of  $\mathbb{R}^n$  and  $\mathbb{C}^n$ . If  $X$  is a metric space with metric  $d$ , one would introduce the set  $\dot{X}$ , consisting of fluxions taking values in  $X$ , which would be defined as equivalence classes of sequences in  $X^{\mathbb{N}}$  under eventual equality. The object  $\dot{X}$  thus produced is not in general a ring, but that is of no consequence. The main point is that the metric  $d$  on  $X$  extends to elements of  $\dot{X}$ , and thus one could define, for example, convergence of a fluxion  $x \in \dot{X}$  to a point  $a \in X$  to occur if and only if  $d(x, a)$  is an infinitesimal real fluxion. The definitions of topological notions in  $X$

would then take precisely the same form as those given in this section. We leave the details to the interested reader.

## 5. POSTSCRIPT

Let us justify the description of fluxions as germs of real-valued sequences.

Regard the set  $\mathbb{N}$  of natural numbers as a topological space with the cofinite topology. In this topology, the open sets are precisely the complements of finite sets. Let  $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$  be the one-point compactification of  $\mathbb{N}$ . The neighborhoods of  $\infty$  in  $\widehat{\mathbb{N}}$  are the subsets of the form  $E \cup \{\infty\}$ , where  $E$  is the complement of a finite subset of  $\mathbb{N}$ .

Now recall a standard general construction in topology. Given a point  $x$  of a topological space  $X$ , and two maps  $f, g: X \rightarrow Y$  (where  $Y$  is any set), then  $f$  and  $g$  define the same germ at  $x$  if there is a neighborhood  $U$  of  $x$  such that the restrictions of  $f$  and  $g$  to  $U$  coincide. Defining the same germ at  $x$  gives an equivalence relation on the space  $Y^X$  of functions from  $X$  into  $Y$ . The resulting equivalence classes are called *germs*.

Thus we see that the ring  $\dot{\mathbb{R}}$  may be identified with the set of germs of real-valued sequences (extended to  $\widehat{\mathbb{N}}$ ) at infinity.

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